3.28. From Formal Language to DNF: The 3D Method

Disjunction Normal Form (DNF) forms a special sub-family of our formal language: those formal sentences where neither disjunctions nor conjunctions are part of a negation, and where disjunctions are not part of a conjunction. (In the language of "scope," that means: in DNF sentences tilde always has smaller scope than wedge, and wedge has smaller scope than vel.) We found that every possible truth table has a matching DNF sentence. Now since every sentence in the formal language (whether DNF or not) has a matching truth table, and every truth table has a matching DNF sentence, it follows that every sentence in the formal language has a matching DNF sentence.

But if we're given a formal sentence not in DNF, and asked to find a matching DNF sentence, how should we go about finding that DNF sentence? Our only recourse so far is to take a detour through truth tables: build the truth table for the original sentence, and then build a DNF sentence matching that truth table.[1]

Here we develop a second, quite different method for matching non-DNF sentences with their DNF counterparts – a method which bypasses truth tables altogether. And while it might seem a mere formal curiosity, in fact this technique will have interesting applications in discussion of later topics.

The matching procedure consists of rules for rewriting any formal sentence not already in Disjunctive Normal Form. And each 'rewrite rule' in this procedure is an application of an argument form recognized as valid.

[1] We follow the general procedure for building a truth table for any formal sentence (spelled out in "*3.16. Construction Meets Semantics*") and the procedure for matching any truth table with a DNF sentence (from "*3.27. Valuation Disjunctions, Expressive Adequacy, and Disjunctive Normal Form*," Section 3).

The Chapter Three formal language allows the following sorts of sentences.

1. Sentence Letters
2. Negations of Sentence Letters
3. Negations of Negations
4. Negations of Conjunctions
5. Negations of Disjunctions
6. Conjunctions of Sentence Letters
7. Conjunctions of Negations
8. Conjunctions of Conjunctions
9. Conjunctions of Disjunctions
10. Disjunctions of Sentence Letters
11. Disjunction of Negations
12. Disjunctions of Conjunctions
13. Disjunctions of Disjunctions

DNF rules out any sentence which is (or has a part which is) of types 3, 4, 5, or 9. (All other formal sentences are already in DNF.) Our sentence 'rewrite' rules are devoted to replacing any occurrence of these four offending types of sentences with equivalents in DNF.

1. Double Negation. The first rewrite rule employs an argument form traditionally called "**Double Negation**".

Double Negation

$$\frac{\sim\sim\bullet}{\bullet} \qquad\qquad \frac{\bullet}{\sim\sim\bullet}$$

Both formal semantics and informal English intuitions confirm that, for instance, "It rained yesterday" follows validly from "It did not fail to rain yesterday," and vice versa.

Our interest here is only in the left form of argument, which clears off a pair of tildes. The validity of this inference holds when the whole sentence is a double negation.

$$\frac{\sim\sim(P \wedge Q)}{(P \wedge Q)}$$

And it holds equally validly when only *part* of the original is a double negation.

$$\frac{(P \wedge \sim\sim Q)}{(P \wedge Q)}$$

So the Double Negation rewrite rule will remove any pair of tildes – either at the left of the whole sentence, or inside that sentence. All of the following are thus examples of the Double Negation rule.

$\underline{\sim\sim P} \;\Rightarrow\; P$

$\underline{\sim\sim(P \wedge Q)} \;\Rightarrow\; (P \wedge Q)$

$(P \vee \underline{\sim\sim Q}) \;\Rightarrow\; (P \vee \sim\sim Q)$

$\sim((P \vee Q) \wedge \underline{\sim\sim(R \vee S)}) \;\Rightarrow\; \sim((P \vee Q) \wedge (R \vee S))$

This rule handles the first sort of non-DNF sentence on our list.

3. Negations of Negations

Note that by itself this rule doesn’t guarantee a DNF output. In the last example, Double Negation transformed “$\sim((P \vee Q) \wedge \underline{\sim\sim(R \vee S)})$” into “$\sim((P \vee Q) \wedge (R \vee S))$”; but “$\sim((P \vee Q) \wedge (R \vee S))$” isn’t a DNF sentence. (It’s a negation of a conjunction, which is our next sort of non-DNF sentence.) But our next two rewrite rules will close that gap.

2. DeMorgan's Law. The second rewrite rule is based on valid forms of inference noted earlier under the name "**DeMorgan's Law**".[2]

DeMorgan's Law

$$\frac{\sim(\bullet \wedge \blacktriangle)}{(\sim\bullet \vee \blacktriangle)} \qquad\qquad \frac{\sim(\bullet \vee \blacktriangle)}{(\sim\bullet \wedge \sim\blacktriangle)}$$

$$\frac{(\sim\bullet \wedge \sim\blacktriangle)}{\sim(\bullet \vee \blacktriangle)} \qquad\qquad \frac{(\sim\bullet \vee \sim\blacktriangle)}{\sim(\bullet \wedge \blacktriangle)}$$

Again we're only interested in half of these patterns – here, the upper two, which push an outer tilde in to the two parts of a conjunction or disjunction. Each holds validly when the entire sentence is a negated conjunction or disjunction.

$$\frac{\sim(P \wedge Q)}{(\sim P \vee \sim Q)} \qquad\qquad \frac{\sim(P \vee Q)}{(\sim P \vee \sim Q)}$$

But the inference is equally valid when only part of the original sentence is a negated conjunction or disjunction.

$$\frac{(R \wedge \underline{\sim(P \wedge Q)})}{(R \wedge (\sim P \vee \sim Q))} \qquad\qquad \frac{(S \vee \sim(P \vee Q))}{(S \vee \underline{(\sim P \wedge \sim Q)})}$$

So whenever a formal sentence, or any part of that sentence, contains a negated conjunction or disjunction, we rewrite it using DeMorgan's Law.

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[2] In "*3.17. Semantic Concepts*".

The following are some examples of the rewrite rule at work.

$$\underline{\sim(P \land Q)} \;\Rightarrow\; \underline{(\sim P \lor \sim Q)}$$
$$\sim(R \land \underline{\sim\sim(P \land Q)}) \;\Rightarrow\; \sim(R \land \underline{(\sim P \lor \sim Q)})$$
$$((R \lor S) \land \underline{\sim\sim(P \lor Q)}) \;\Rightarrow\; ((R \lor S) \land \underline{(\sim P \land \sim Q)})$$

DeMorgan’s Law addresses two more non-DNF sentences from our list.

4. Negations of Conjunctions
5. Negations of Disjunction

Once again, this rewrite rule by itself will not always yield a sentence in DNF. In the second and third examples above, the output – “$\sim(R \land (\sim P \lor \sim Q))$” and “$((R \lor S) \land \underline{(\sim P \lor \sim Q)})$” – is not a sentence in DNF. Only the three rewrite rules, taken together, ensure a DNF sentence at the end of the rewriting.

DeMorgan’s Law, as applied in our rewrite rules, has the effect of forcing the tilde to have a **smaller scope** than the wedge or vel. Starting with the negation of a conjunction or disjunction (where the tilde has wider scope), we end with a conjunction or disjunction of negations (where the two tildes each have smaller scope than the wedge or vel).

3. Distribution. The last rewrite rule addresses the remaining non-DNF sentence from our original list.

9. Conjunctions of Disjunctions

Here we appeal to a valid argument form earlier labeled “**Distribution**”.[3]

[3] In “*3.10. Scope*,” Section 2.

Distribution

$$\frac{(\underline{\bullet \wedge} (\blacktriangle \vee \heartsuit))}{((\underline{\bullet \wedge} \blacktriangle) \vee (\underline{\bullet \wedge} \heartsuit))} \qquad \frac{(\underline{\bullet \vee} (\blacktriangle \wedge \heartsuit))}{((\underline{\bullet \vee} \blacktriangle) \wedge (\underline{\bullet \vee} \heartsuit))}$$

$$\frac{((\underline{\bullet \wedge} \blacktriangle) \vee (\underline{\bullet \wedge} \heartsuit))}{(\underline{\bullet \wedge} (\blacktriangle \vee \heartsuit))} \qquad \frac{((\underline{\bullet \vee} \blacktriangle) \wedge (\underline{\bullet \vee} \heartsuit))}{(\underline{\bullet \vee} (\blacktriangle \wedge \heartsuit))}$$

We're only interested here in the top left inference pattern – yielding rewrite rules illustrated by the following examples.

$$(\underline{P \wedge} (Q \vee R)) \;\Rightarrow\; ((\underline{P \wedge} Q) \vee (\underline{P \wedge} R))$$
$$(\underline{\sim(P \wedge Q) \wedge} (R \vee {\sim}S) \;\Rightarrow\; (\; (\underline{\sim(P \wedge Q) \wedge} R) \vee (\underline{\sim(P \wedge Q) \wedge} S)\;)$$

Once again, that last example of distribution yields a sentence – "$((\underline{\sim(P \wedge Q) \vee} R) \wedge (\underline{\sim(P \wedge Q) \vee} S))$" – which is not in DNF. Only the combination of all three rules guarantees a DNF outcome.

Distribution as used here has the effect of switching the size of the **scope** of the wedge and vel: starting with a sentence where the wedge has wider scope, we end with an equivalent sentence where the vel has wider scope.

4. The 3D Method. The combination of these three rewrite rules – Double Negation, DeMorgan's Law, and Distribution – forms a rewrite system called (for obvious reasons) "**The 3D Method**". The 3D Method is guaranteed to transform any non-DNF sentence of the Chapter Three language into Disjunctive Normal Form.

3D Method

Given a non-DNF sentence, we rewrite it (including any non-DNF parts) according to the following rules.

Double Negation (DN)

$$\sim\sim\bullet \;\Rightarrow\; \bullet$$

DeMorgan’s Law (DM)

$$\sim(\bullet \wedge \blacktriangle) \;\Rightarrow\; (\sim\bullet \vee \sim\blacktriangle)$$
$$\sim(\bullet \vee \blacktriangle) \;\Rightarrow\; (\sim\bullet \wedge \sim\blacktriangle)$$

Distribution (D)

$$(\underline{\bullet \wedge} (\blacktriangle \vee \heartsuit)) \;\Rightarrow\; ((\underline{\bullet \wedge} \blacktriangle) \vee (\underline{\bullet \wedge} \heartsuit))$$

To apply the 3D Method, we apply DN, DM, and D to ensure that each vel has a wider scope than any wedge or tilde, and that each wedge has a wider scope than any tilde. Returning to our earlier list of non-DNF sentences makes clear how each of the rewrite rules is to be applied.

Chapter Three Formal Sentences

1. Sentence Letters

2. Negations of Sentence Letters
3. Negations of Negations
4. Negations of Conjunctions
5. Negations of Disjunctions

6. Conjunctions of Sentence Letters
7. Conjunctions of Negations
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9. Conjunctions of Disjunctions

10. Disjunctions of Sentence Letters
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12. Disjunctions of Conjunctions
13. Disjunctions of Disjunctions

Non-DNF Sentence	**Rewrite Rule**
3. Negation of Negation	**DN**
4. Negation of Conjunction	**DM**
5. Negation of Disjunction	**DM**
9. Conjunction of Disjunction(s)	**D**

The following sentence provides an illustration. The sentence is not in Disjunctive Normal Form, since the main connective of the sentence (hence the connective having the widest scope) is the leftmost tilde.

1. ~((P ∧ Q) ∨ (~~R ∧ S))

Since the sentence is a **negation of a disjunction**, we apply **DeMorgan's Law** to force the leftmost tilde to take smaller scope.

1. ~((P ∧ Q) ∨ (~~R ∧ S))
2. (~(P ∧ Q) ∧ ~(~~R ∧ S)) 1, DM

The left part of this conjunction, "~(P ∧ Q)," is not in DNF, since it is a **negation of a conjunction**. Another application of **DeMorgan's Law** forces the tilde to take smaller scope.

2. (<u>~(P ∧ Q)</u> ∧ ~(~~R ∧ S))
3. (<u>(~P ∨ ~Q)</u> ∧ ~(~~R ∧ S)) 2, DM

The sentence "~(~~R ∧ S)" is likewise a **negation of a conjunction**, calling for a further application of **DeMorgan's Law**.

3. ((~P ∨ ~Q) ∧ <u>~(~~R ∧ S)</u>)
4. ((~P ∨ ~Q) ∧ (~~~R ∨ ~S)) 3, DM

“~~~R” is a **negation of a negation**. This calls for **Double Negation**.

4. $((\sim P \lor \sim Q) \land (\underline{\sim\sim\sim R} \lor \sim S))$
5. $((\sim P \lor \sim Q) \land (\underline{\sim R} \lor \sim S))$ 4, DN

Finally, the wedge here has larger scope than either vel, making this sentence a **conjunction of disjunctions**. **Distribution** gives the right vel wider scope than any wedge.

5. $(\underline{(\sim P \lor \sim Q) \land} (\sim R \lor \sim S))$
6. $(\ (\underline{(\sim P \lor \sim Q) \land} \sim R) \lor (\underline{(\sim P \lor \sim Q) \land} \sim S) \ )$ 5, D

Each of the two conjunctions – “((~P ∨ ~Q) ∧ ~R)” and “((~P ∨ ~Q) ∧ ~S)” – is still a conjunction with a disjunction as part, hence a **conjunction of disjunction(s)**. Two further applications of **Distribution** ensure than no remaining vel has a smaller scope than a wedge.

6. $(\ ((\sim P \lor \sim Q) \underline{\land \sim R}) \lor ((\sim P \lor \sim Q) \land \sim S) \ )$
7. $(\ ((\sim P \underline{\land \sim R}) \lor (\sim Q \underline{\land \sim R})) \lor ((\sim P \lor \sim Q) \land \sim S) \ )$ 6, D

7. $(\ ((\sim P \land \sim R) \lor (\sim Q \land \sim R)) \lor ((\sim P \lor \sim Q) \underline{\land \sim S}) \ )$
8. $(\ ((\sim P \land \sim R) \lor (\sim Q \land \sim R)) \lor ((\sim P \underline{\land \sim S}) \lor (\sim Q \land \sim S)) \ )$ 7, D

Sentence (8) is in Disjunctive Normal Form. Removing unnecessary parentheses makes for a more readable result.

8. $(\sim P \land \sim R) \ \lor \ (\sim Q \land \sim R) \ \lor \ (\sim P \land \sim S) \ \lor \ (\sim Q \land \sim S)$

Summary: The 3D Method

To transform any non-DNF sentence into Disjunctive Normal Form, rewrite it (including any non-DNF parts) by applying the following rules.

Double Negation (DN)

$$\sim\sim\bullet \;\Rightarrow\; \bullet$$

DeMorgan's Law (DM)

$$\sim(\bullet \wedge \blacktriangle) \;\Rightarrow\; (\sim\bullet \vee \sim\blacktriangle)$$

$$\sim(\bullet \vee \blacktriangle) \;\Rightarrow\; (\sim\bullet \wedge \sim\blacktriangle)$$

Distribution (D)

$$(\underline{\bullet \wedge} (\blacktriangle \vee \heartsuit)) \;\Rightarrow\; ((\underline{\bullet \wedge} \blacktriangle) \vee (\underline{\bullet \wedge} \heartsuit))$$

Non-DNF Sentence	**Rewrite Rule**
Negation of Negation	**DN**
Negation of Conjunction	**DM**
Negation of Disjunction	**DM**
Conjunction of Disjunction(s)	**D**